

Bosonization in arbitrary dimensions

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(February 1, 2008)

Using methods of functional integration, and performing simple Gaussian integrals, I show that an interacting system of electrons can be bosonized in arbitrary dimensions, in terms of the electrostatic potential which mediates the interaction between them. Working with the bosonic field, the system is shown to exhibit localized structures reminiscent of striping in the cuprates.

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Attempts at bosonizing a gas of interacting fermions date back to Tomonaga¹, in the context of superconductivity. Luttinger² followed with an exact solution of a model Hamiltonian in one dimension. Lieb and Mattis³ perfected Luttinger's solution shortly thereafter. Luther⁴ and Haldane⁵ published seminal papers which extended the older one-dimensional methods to higher dimensions. Their ideas have been developed in recent years by Houghton et al⁶, Feldman et al⁷, Castro-Neto and Fradkin⁸, and Frohlich and Gotschmann⁹. Other theoretical methods involving techniques to bosonize fermions in one or two dimensions have also appeared in the literature¹⁰⁻¹⁴. These ideas almost exclusively entail the use of fermionic creation and annihilation operators, and as such are difficult to apply in greater than one spatial dimension.

One of the chief motivations for bosonizing a system of interacting electrons is to investigate the possibility of these electrons forming bound states which may not be accessible via the usual perturbative methods. Recently, a Luttinger liquid has been invoked by Anderson¹⁵ in an effort to understand high temperature superconductivity (HTSC). We will provide in this paper a simpler, exact alternative to the Luther-Haldane formalism. The main focus of this paper will be to establish a straightforward way to replace a system of interacting electrons with an equivalent system of interacting bosons using functional integral methods. In this paper it will be shown that bosonization of a certain kind can always be applied exactly to interacting fermionic systems in arbitrary dimensions. The ideas established in this paper may lead to insight into the mechanism of high temperature superconductivity, especially as the detailed physics of the HTSC compounds is included into the problem. This paper should also be applicable to strongly correlated electron systems in general. It should also be relevant to the study of striping in the cuprates. In fact we will provide later in the paper a qualitative model for this phenomenon of striping.

Bosonization has attracted attention in work on relativistic field theories as well. In this paper I shall carry out in the non-relativistic regime appropriate to condensed matter physics, the program of bosonization developed recently by Bannerjee in relativistic Quantum

Field Theory (QFT)^{16,17}. Bannerjee used the method of functional integration to eliminate matter fields in the standard expression for the action, leaving just bosonic fields for consideration. The focus in this paper will be on phenomena which emerge when these methods are applied to issues in condensed matter physics for $d \geq 1$.

Our technique is related closely to the work of Fogedby¹⁸, Lee and Chen¹⁹ and Kopietz²⁰. The focus in these papers has been the computation of the electronic Green's function, in an effort to understand the origin of non-Fermi-liquid behavior. I shall instead concentrate on uncovering physics which lies in the effective action for the bosonic fields that replace the fermionic fields. As indicated above, I obtain a tentative model for striping in the cuprates. This paper shows how to eliminate the approximations invoked by Kopietz to carry out successfully the program of bosonization in $d > 1$. Last but not least, the techniques utilized here will be seen to be more economical than in previous papers, as a result of which the final, novel results are more transparent. Rather than diminishing previous results, it is hoped that the perspectives of this paper can be combined with existing techniques to shed more light on HTSC.

Without further ado, I note that the usual partition function for a gas of interacting electrons (in atomic units; $\hbar = c = e = m = 1$) is given by

$$\begin{aligned} Q &= \int \mathcal{D}\psi^* \mathcal{D}\psi \exp iS \\ S &= \int dt \int d^d x \psi^*(\vec{x}, t) (i\partial_t - H_0) \psi(\vec{x}, t) \\ &\quad - \int dt \int d^d x \psi^*(\vec{x}, t) \int d^d x' |\psi(\vec{x}', t)|^2 |\vec{x} - \vec{x}'|^{-1} \psi(\vec{x}, t) \\ H_0 &= -\frac{1}{2} \nabla^2 \end{aligned} \quad (1)$$

where ψ and ψ^* are two-vectors describing fermion fields having a spin 1/2. H_0 can be modified with an extra term describing the interaction of the electrons with an underlying lattice. This partition function can be derived by introducing the scalar electrostatic potential:

$$Q \sim \int \mathcal{D}\psi^* \mathcal{D}\psi \mathcal{D}\phi \exp iS'$$

$$S' = \int dt \int d^d x \left[\psi^* (i\partial_t - \phi) \psi - \psi^* H_0 \psi - \frac{|\vec{\nabla}\phi|^2}{8\pi} \right] \quad (2)$$

where an irrelevant normalization constant has been ignored.

Integrating out the electrostatic potential using Gaussian integration in 2 leads directly to 1. On the other hand, if the matter fields ψ and ψ^* are integrated out in 2, the partition function involves just the electrostatic field ϕ . In general, the electromagnetic vector potential \vec{A} should also be included. Traditionally this is not done in condensed matter physics on grounds that it is the Coulomb interaction which dominates the physics. Elimination of the matter fields then leads to:

$$Q \sim \int \mathcal{D}\phi \exp i \int dt \int d^d x \left\{ -\frac{|\vec{\nabla}\phi|^2}{8\pi} \right\} \times (\det [-i(\partial_t - \phi + \mu - H)])^{-1} \quad (3)$$

where μ signifies a constraint used to guarantee number conservation, since the integration over the matter fields includes cases involving broken symmetry. Note that this constraint is formally equivalent to an interaction of the electrons with a uniform background field. The way μ has been defined, $\mu > 0$ implies an interaction with a uniform positive background. We see here an analogy with the Wigner-Seitz model. An increasing number density of electrons is signalled by an increasing μ . The appearance of the scalar field ϕ in 3 implies that we are dealing with a spin-zero object, but it should be remembered that ϕ is an approximation to the photonic field, which carries a unit spin. Given that the gauge degree of freedom represented by ϕ can be absorbed into the electronic wavefunction, we see that there is a certain similarity between our approach and that of Haldane⁵ who performed a decomposition of the electronic wavefunction in terms of its phase. I will now evaluate the determinant in 3, which contains non-trivial physics. This evaluation is more convenient in Euclidean space, which entails setting $t \rightarrow -it$. I then need the determinant of the operator $(\partial_t + \phi - \mu + H_0)$. Upon considering an associated *heat diffusion equation* viz., $-\partial_\tau g(\vec{x} - \vec{x}', t - t', \tau) = (\partial_t + \phi - \mu + H_0) g(\vec{x} - \vec{x}', t - t', \tau)$ the determinant is obtained as $\exp -\zeta'(0)$ ²¹, where

$$\zeta(s) = \frac{1}{\Gamma(s)} \int d\tau \tau^{s-1} \int dt \int d^3x g(\vec{0}, 0, \tau) \quad (4)$$

Here attention has been restricted to three dimensions for ease of presentation. Our method of evaluating determinants of operators is fairly well-established²¹. The basic idea is to perform the evaluation formally with a ϕ which is constant, and then to use the *form* obtained thusly to write down the complete expression for the effective potential. The manipulations of this equation are fairly straightforward, and the only care one needs to exercise is in taking Laplace transforms, considering only

those frequency domains which ensure convergence of the overall expression. The effective potential which arises from the determinant is given by:

$$V_{eff}(\phi - \mu) = (|\phi - \mu|)^{5/2} \times [\Psi(-5/2) - \Gamma(-5/2) \ln(|\phi - \mu|)] \quad (5)$$

where Ψ in 5 is a digamma function and should not be confused the fermionic field used earlier. Given that we have here a nonrelativistic formulation of electron dynamics, it is seen that temporal derivatives do not appear in the bosonized version of the partition function. ϕ refers to the (time-independent) electrostatic potential. As such the partition function involving the field ϕ can be redefined with a delta-function, in much the same way as gauge constraints are imposed on path-integral formulations. On classical grounds only $\phi > 0$ needs to be considered, since we have a system of interacting like charges. More generally, if we are considering quantum effects, negative values of ϕ could occur. It can be shown graphically that for sufficiently small values of μ , the effective potential displays a single minimum for $\phi > 0$. There is of course a second minimum for negative values of the field. I will revisit this double-well form of V_{eff} shortly. The compact form for V_{eff} was obtained relatively economically. Besides the ease of derivation, the simple form for V_{eff} displays a double-well structure which will shortly yield some novel insights into the interacting electron system we started out with.

If the potential is expanded around the positive minimum which occurs for sufficiently low values of the chemical potential, then the Euler-Lagrange equation associated with the effective Lagrangian can be written as:

$$(\nabla^2 - \lambda^{-2}) \delta\phi = 0 \\ \lambda^{-2} = 4\pi V_{eff}''(\phi_{min}) \quad (6)$$

This is the analog of Laplace's equation for a system interacting via a *screened* Coulomb interaction. The right hand side of 6 signifies that the field configuration is such that there are no free charges in the system. I interpret this as an insulating phase! The reason for restricting attention to positive values of the field is that the Euler-Lagrange equation represents the classical limit. The basic assumption behind this argument is that the system settles into a state described by ϕ_{min} and harmonic perturbations around it.

More complicated situations than the one described by 6 can also occur. The previous discussion regarding the insulating regime can be expanded by considering the following double-well representation for the effective potential, viz., $V_{eff} = V_o - (\alpha/2)(\phi - \mu)^2 + (\beta/4)(\phi - \mu)^4$. The two minima for this function occur at $\phi_{min} = \mu \pm \sqrt{\alpha/\beta}$. From this it follows that the Euler equation is:

$$\nabla_\xi^2 \tilde{\phi} + \tilde{\phi} - \left(\frac{\beta}{\alpha}\right) \tilde{\phi}^3 = 0 \quad (7)$$

where $\tilde{\phi} = \phi - \mu$, and $\vec{\xi} = \sqrt{4\pi\alpha} \vec{x}$. Note that for μ sufficiently small, there is only one minimum for $\phi > 0$. In one dimension, as $x \rightarrow 0$, $\phi \approx (1/2)x^2(-\mu + (\beta/\alpha)\mu^3)$, while for $x \rightarrow \pm\infty$, $\phi \sim \phi_{min}$. This allows us to connect neighboring insulating regions via a domain wall. Going to a cylindrical geometry, it is possible to show approximately that as the cylindrical co-ordinate $\rho \rightarrow 0$, $\phi - \mu \sim \rho^s$, with $s=0,1,2,\dots$. While for $\rho \rightarrow \infty$, $\phi - \mu \sim \phi_{min} [1 - s^2/(2\rho^2)]$. By the arguments given above, the domain wall between the defect and the surrounding insulating phase has free charges flowing through it. An array of such localized defects are reminiscent of the striping which occurs in the cuprates. The model we just described is complementary to the approach of Castro-Neto et al¹³. In spherical geometry, as the radial co-ordinate $r \rightarrow 0$, $\phi - \mu \sim r^l$, with $l=0,1,2,\dots$. For distances far from the center of such spherulitic defects, $\phi - \mu \sim \phi_{min} [1 - l(l+1)/(2r^2)]$. This approximate treatment in non-planar geometry can be trivially generalized using the unabridged form of V_{eff} .

As μ increases, both minima in V_{eff} occur for positive values of the field ϕ . As indicated below 3, the appearance of μ is formally equivalent to introducing a Wigner-Seitz positive uniform background model. We would therefore like to argue that for sufficiently large values of μ , the system behaves as a metal. We currently cannot describe this transition from an insulating to a metallic phase. The only remark we can make is the following: Given that classically the system can settle into one of two energy minima, it must be possible to have spatial domains in which the system has $\phi = \pm\phi_{min}$, with domain walls permitting a transition from one state to the other. For example, if we assume that the three-dimensional system exhibits variation in only one direction, then the domain wall can be described by $\phi = \mu + \phi_{min} \tanh[x\sqrt{2\pi\alpha}]$. This domain wall has free charges flowing through it, i.e., it satisfies 6, but with a source term on the right hand side. But this is similar to the defects discussed above, in the insulating phase. It is only if there are a large number of these domain walls, and they form a connected network, will the system conduct appreciably.

We are currently in the process of computing the elec-

tronic Green's function, given that the system of interacting electrons can be bosonized.

I would like to acknowledge electronic correspondence with P.W. Anderson and P. Kopietz rectifying gaps in my knowledge of previous literature.

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- ¹ Sin-itiro Tomonaga, Prog. Theo. Phys. **5**, 544 (1950).
 - ² J.M. Luttinger, J. Math. Phys., **4**, 1154 (1963).
 - ³ D.C. Mattis, E.H. Lieb, J. Math. Phys., **6**, 304, (1965).
 - ⁴ A. Luther, Phys. Rev. B **19**, 320 (1979).
 - ⁵ F.D.M. Haldane, J. Phys. C **14**, 2585 (1981); F.D.M. Haldane, Varenna Lecture Notes, (1992).
 - ⁶ A. Houghten and J.B. Marsden, Phys. Rev. B **48**, 7790 (1993).
 - ⁷ J. Feldman, J. Magnan, V. Rivasseau and E. Trubowitz, Europhys. Lett. **24**, 437 (1993); **24**, 521 (1993).
 - ⁸ A.H. Castro-Neto and E. Fradkin, Phys. Rev. B **49**, 10877 (1994).
 - ⁹ J. Frohlich and R. Gotschmann, Phys. Rev. B **55**, 6788 (1997).
 - ¹⁰ T. Martin and D. Loss, Int. J. Mod. Phys. **9**, 495 (1995).
 - ¹¹ O. Heinonen and S. Eggert, Phys. Rev. Lett. **77**, 358 (1996).
 - ¹² A. Georges, G. Kotliar, W. Krauth, M. Rozenberg, Rev. Mod. Phys. **68**, 13 (1996).
 - ¹³ A.H. Castro Neto, C. de C. Chamon, and C. Nayak, Phys. Rev. Lett. **79**, 4629 (1997).
 - ¹⁴ G. Seibold and E. Sigmund, cond-mat/9801135 (1998).
 - ¹⁵ P.W. Anderson, *The theory of superconductivity in the High T_c cuprates*, Princeton E.P., Princeton, N.J. (1997).
 - ¹⁶ R. Bannerjee, Nucl. Phys. B **465** 157 (1996).
 - ¹⁷ R. Bannerjee, Phys. Rev. Lett. **80** 238 (1998).
 - ¹⁸ H.C. Fogedby, J. Phys. C **9**, 3757 (1976).
 - ¹⁹ D.K.K. Lee and Y. Chen, J. Phys. A **21**, 4155 (1988).
 - ²⁰ P. Kopietz and K. Schonhammer, Zeit. fur Phys. B **100**, 259 (1996); P. Kopietz and G.E. Castella, Phys. Rev. Lett. **76**, 4777 (1996).
 - ²¹ P. Ramond, *Field Theory: A modern Primer*, The Benjamin Cummings Pub. Co., Reading, MA (1981).
 - ²² E. Lieb, F.Y. Wu, Phys. Rev. Lett. **20** 1445 (1968).